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Harmonic spinors on semisimple symmetric spaces

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Abstract

Let G/H be a semisimple symmetric space. We consider a Dirac operator D on G/H twisted by a finite dimensional H-representation. We give an explicit integral formula for certain solutions of the equation $D = 0$. In particular, some quotients of standard principal series representations are seen to occur in the kernel of D.
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1. Introduction

Discrete series representations of semisimple Lie groups occur as L^2-spaces of solutions to certain twisted Dirac operators on riemannian symmetric spaces G/K [1,9]. A generalization of this fact is given in [11]. In this article, we consider twisted Dirac operators on nonriemannian symmetric spaces G/H. Under the condition that G and H have equal rank we show that the space of smooth solutions to the twisted Dirac equation is nonzero. In addition, we give an integral formula representing solutions. The integral formula is very similar to the classical Poisson integral representation of harmonic functions, and its generalizations.

Our integral formula for the solutions to the Dirac equation is in the form of a G-intertwining operator from a principal series representation into the sections of a

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twisted spin bundle on G/H. This operator is constructed in a way similar to the construction of intertwining maps into Dolbeault cohomolgy representations (occurring on open orbits in flag varieties), see [2–4]. In that situation the image of the intertwining operator is annihilated by $\bar{\partial}$ and $\bar{\partial}^*$, thus strongly harmonic forms are produced. The results in this article differ considerably in that no complex structure on G/H plays a role, and in fact no invariant complex structure exists in general.

Our results should be considered as the opposite extreme from the results mentioned above for discrete series on G/K in the following sense. Our intertwining operator tells us nothing in the case of $H = K$. Indeed, we reduce to this case. It should be noted that the techniques in the case of $H = K$ are quite different from ours. In particular, for $H \neq K$ the Dirac operator is not elliptic and L^2 techniques are not available.

This article is organized as follows. Section 2 gives some notation and describes the groups which will arise. In Sections 3 and 4, we review the material about spin representations which we will need, define the Dirac operator and compute its square. Section 5 gives a construction of the intertwining operator. In Section 6, we prove that the image of the intertwining operator lies in the kernel of the Dirac operator.

2. Preliminaries

Let G be a connected linear semisimple Lie group with Lie algebra denoted by $\mathfrak{g}$. Let σ be an involution of G and set

$$H = (G^\sigma)_0,$$

the identity component of the fixed point group of σ. Then the homogeneous space G/H is semisimple symmetric, not necessarily riemannian. With respect to the involution σ, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$. Fix a Cartan involution θ commuting with σ and let $K = G^\theta$ be the fixed point group of θ. This induces a Cartan involution of $\mathfrak{g}$, which we also denote by θ, and a Cartan decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s},$$

where $\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. There is a further decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{s}) \oplus (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{s}).$$

We shall assume that the complex rank of G is equal to the complex rank of H:

$$rank(G) = rank(H). \tag{2.1}$$

In particular, one can choose a Cartan subalgebra of $\mathfrak{g}$ in $\mathfrak{h}$. More precisely, let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{h}\cap\mathfrak{s}$. From (2.1) one can choose $\mathfrak{t}$ in $\mathfrak{h}$ so that $\mathfrak{a}\oplus\mathfrak{t}$ is a σ-stable Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{h}$. This gives a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{m}\oplus\mathfrak{a}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{t}$ is a compact Cartan subalgebra of $\mathfrak{m}$.

Now let $\Sigma(\mathfrak{g},\mathfrak{a})$ be the set of restricted roots of $\mathfrak{g}$ relative to $\mathfrak{a}$. As usual, after a choice of positive roots $\Sigma^+(\mathfrak{g},\mathfrak{a})$, one defines the nilpotent subalgebra

$$\mathfrak{n} = \sum_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the restricted root space corresponding to α. Denote by M_0, A and N the analytic subgroups of G with Lie algebra $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then one obtains a parabolic subgroup P of G with Langlands decomposition

$$P = MAN,$$

where M is the closed subgroup $Z_K(\mathfrak{a})M_0$ of G, with $Z_K(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in K. As M contains a compact Cartan subgroup, P is cuspidal.

It is crucial for our construction that $\mathfrak{a}$ is maximal abelian in $\mathfrak{h}\cap\mathfrak{s}$. Several simple consequences of this, which we will refer to later, are contained in the following lemma.

Lemma 2.2. *Assume $\mathfrak{a}$ and $P = MAN$ are as above. Then the following statements hold*:

(a) *$P\cap H$ is a minimal parabolic subgroup of H and*

$$P\cap H = (M\cap H)A(N\cap H).$$

(b) *The orbit $H\cdot eP$ of H in G/P is a closed orbit and*

$$H\cdot eP \simeq H/H\cap P \simeq H\cap K/H\cap M.$$

(c) *$M, H\cap M$ and $K\cap M$ all have the same (complex) rank.*

(d) $\mathfrak{q} = (\mathfrak{m}\cap\mathfrak{q})\oplus(\mathfrak{n}\cap\mathfrak{q})\oplus(\bar{\mathfrak{n}}\cap\mathfrak{q})$ *and* $\mathfrak{m}\cap\mathfrak{q} = (\mathfrak{m}\cap\mathfrak{s})\oplus(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})$.

Consider the system of roots of $(\mathfrak{a}\oplus\mathfrak{t})^C$ in $\mathfrak{g}^C$, which we denote by $\Delta(\mathfrak{g}^C,(\mathfrak{a}\oplus\mathfrak{t})^C)$. We fix once and for all a positive system in $\Delta(\mathfrak{g}^C,(\mathfrak{a}\oplus\mathfrak{t})^C)$ with the property that the restriction to $\mathfrak{a}$ of a positive root is a positive restricted root. This determines a system of positive roots in $\Delta(\mathfrak{h}^C,(\mathfrak{a}\oplus\mathfrak{t})^C)$, the set of roots of $(\mathfrak{a}\oplus\mathfrak{t})^C$ in $\mathfrak{h}^C$. We will denote by $\rho(\mathfrak{g})$ the half-sum of positive $(\mathfrak{a}\oplus\mathfrak{t})^C$-roots in $\mathfrak{g}^C$. Note that a positive system of $\mathfrak{t}$-roots for $\mathfrak{m}$ may be specified arbitrarily.

Observe that the discrete series of M is nonempty. So let (δ, W) be a discrete series representation of M in some Hilbert space W. Let ν be a linear form on $\mathfrak{a}$, and denote by $\rho_{\mathfrak{g}}$ the half-sum of (restricted) positive roots:

$$\rho_{\mathfrak{g}} = \frac{1}{2} \sum_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \alpha.$$

As usual, one can form the nonunitary principal series representation $\pi_{\delta,v}$ of G as the induced representation $Ind_{MAN}^{G}(\delta \otimes e^{v} \otimes 1)$ of G from the irreducible P-representation $\delta \otimes e^{v} \otimes 1$ on $W \otimes \mathbf{C}_v$ (normalized induction). The induced representation space is the space of smooth sections of the homogeneous vector bundle for $W \otimes \mathbf{C}_{v+\rho_{\mathfrak{g}}}$. This space of sections is denoted by $C^\infty(G/P, \mathscr{W} \otimes \mathbf{C}_{v+\rho_{\mathfrak{g}}})$ and consists of smooth functions $f : G \to W \otimes \mathbf{C}_{v+\rho_{\mathfrak{g}}}$ such that

$$f(gman) = (\delta(m)^{-1} \otimes e^{-(v+\rho_{\mathfrak{g}})(\ln(a))})(f(g)), \tag{2.3}$$

for every $g \in G$ and $man \in P = MAN$. We shall refer to δ and v as the parameters of the nonunitary principal series representation $\pi_{\delta,v}$ of G.

3. The spin representations

We apply the general construction of Clifford algebras and spin representations associated to a vector space with a nondegenerate (possibly indefinite) symmetric bilinear form. A good reference for this construction is [5, Chapter 6]. Note that under our equal rank assumption (2.1) $\mathfrak{q}$ is even dimensional. Let $\langle\ ,\ \rangle_{\mathfrak{q}}$ be the restriction of the killing form of $\mathfrak{g}$ to $\mathfrak{q}$. Extend $\langle\ ,\ \rangle_{\mathfrak{q}}$ linearly to the complexification $\mathfrak{q}^{\mathbf{C}}$ of $\mathfrak{q}$ and use the same symbol $\langle\ ,\ \rangle_{\mathfrak{q}}$ to denote this extension. Let V and V^* be two dual maximal isotropic subspaces of $\mathfrak{q}^{\mathbf{C}}$ relative to $\langle\ ,\ \rangle_{\mathfrak{q}}$. Denote by $\wedge V^*$ the exterior algebra of V^*:

$$\wedge V^* = \bigoplus_{l=0}^{\frac{dim(\mathfrak{q})}{2}} \wedge^l V^*,$$

equipped with the interior product ι and exterior multiplication ε:

$$\iota(v)(v_1^* \wedge \cdots \wedge v_l^*) = \langle v, v_1^* \rangle_{\mathfrak{q}} v_2^* \wedge \cdots \wedge v_l^* - v_1^* \wedge \iota(v)(v_2^* \wedge \cdots \wedge v_l^*),$$

$$\varepsilon(v^*)(v_1^* \wedge \cdots \wedge v_l^*) = v^* \wedge v_1^* \wedge \cdots \wedge v_l^*,$$

for all $v \in V$, $v^* \in V^*$ and $v_1^* \wedge \cdots \wedge v_l^* \in \wedge^l V^*$. On the other hand, the Clifford algebra $Cl(\mathfrak{q}^{\mathbf{C}})$ of $\mathfrak{q}^{\mathbf{C}}$ is defined as the quotient of the tensor algebra $T(\mathfrak{q}^{\mathbf{C}})$ of $\mathfrak{q}^{\mathbf{C}}$ by the ideal $\mathscr{I}$ generated by elements $X \otimes Y + Y \otimes X - \langle X, Y \rangle_{\mathfrak{q}}$, X and Y in $\mathfrak{q}^{\mathbf{C}}$:

$$Cl(\mathfrak{q}^{\mathbf{C}}) = T(\mathfrak{q}^{\mathbf{C}})/\mathscr{I}.$$

Now define a map γ by

$$\gamma : \mathfrak{q}^{\mathbb{C}} \to End(\wedge V^*), \qquad v + v^* \mapsto \iota(v) + \varepsilon(v^*), \tag{3.1}$$

where, as usual, $End(\wedge V^*)$ denotes the vector space of complex linear endomorphisms of $\wedge V^*$. Observing that

$$\gamma(X) \circ \gamma(Y) + \gamma(Y) \circ \gamma(X) = \langle X, Y \rangle_{\mathfrak{q}}, \quad \forall X, Y \in \mathfrak{q}^{\mathbb{C}},$$

one can extend γ naturally to a map $\tilde{\gamma}$ on the Clifford algebra of $\mathfrak{q}^{\mathbb{C}}$ by

$$\tilde{\gamma} : Cl(\mathfrak{q}^{\mathbb{C}}) \to End(\wedge V^*), \qquad X_1 X_2 \cdots X_p \mapsto \gamma(X_1) \circ \gamma(X_1) \circ \cdots \circ \gamma(X_p).$$

Next, let $SO(\mathfrak{q})$ be the identity component of the Lie group of orthogonal endomorphisms of $\mathfrak{q}$ relative to $\langle\ ,\ \rangle_{\mathfrak{q}}$ with Lie algebra

$$\mathfrak{so}(\mathfrak{q}) = \{A \in End(\mathfrak{q}) \mid \langle AX, Y \rangle_{\mathfrak{q}} + \langle X, AY \rangle_{\mathfrak{q}} = 0,\ \forall X, Y \in \mathfrak{q}\}.$$

For all X and Y in $\mathfrak{q}$, define the endomorphism $R_{X,Y}$ by

$$R_{X,Y}(W) = \langle Y, W \rangle_{\mathfrak{q}} X - \langle X, W \rangle_{\mathfrak{q}} Y, \quad W \in \mathfrak{q},$$

and observe that

$$R_{X,Y} \in \mathfrak{so}(\mathfrak{q}), \quad \forall X, Y \in \mathfrak{q}.$$

In fact, these $R_{X,Y}$ generate the Lie algebra $\mathfrak{so}(\mathfrak{q})$. Therefore one can define an injective Lie algebra homomorphism, which embeds $\mathfrak{so}(\mathfrak{q})$ in the Clifford algebra of $\mathfrak{q}^{\mathbb{C}}$ [5, Lemma 6.2.2] by

$$\Phi : \mathfrak{so}(\mathfrak{q}) \to Cl_2(\mathfrak{q}^{\mathbb{C}}), \quad R_{X,Y} \mapsto \tfrac{1}{2}(XY - YX),$$

where $Cl_2(\mathfrak{q}^{\mathbb{C}})$ is the Lie algebra defined as the subspace of $Cl(\mathfrak{q}^{\mathbb{C}})$ generated by the elements $X_1 X_2$, with X_1 and X_2 in $\mathfrak{q}$. Next, let $S^+_{\mathfrak{q}}$ and $S^-_{\mathfrak{q}}$ be the following vector subspaces of $\wedge V^*$:

$$S^+_{\mathfrak{q}} = \bigoplus_{l\ \text{even}} \wedge^l V^* \quad \text{and} \quad S^-_{\mathfrak{q}} = \bigoplus_{l\ \text{odd}} \wedge^l V^*.$$

Obviously, both $S^+_{\mathfrak{q}}$ and $S^-_{\mathfrak{q}}$ are invariant under the $\tilde{\gamma}$-action of $Cl_2(\mathfrak{q}^{\mathbb{C}})$:

$$\tilde{\gamma}(a)(S^{\pm}_{\mathfrak{q}}) \subset S^{\pm}_{\mathfrak{q}}, \quad \forall a \in Cl_2(\mathfrak{q}^{\mathbb{C}}).$$

Next, consider the two involutions α and $*$ in $Cl(\mathfrak{q})$ defined by

$$\alpha(\gamma(v_1) \cdots \gamma(v_k)) = (-1)^k \gamma(v_1) \cdots \gamma(v_k), \quad \forall v_j \in \mathfrak{q}$$

and

$$(\gamma(v_1)\cdots\gamma(v_k))^* = (-1)^k\gamma(v_k)\cdots\gamma(v_1), \quad \forall v_j \in \mathfrak{q}.$$

The spin group of $\mathfrak{q}$ with respect to $\langle\ ,\ \rangle_{\mathfrak{q}}$ is the subset of the Clifford algebra $Cl(\mathfrak{q})$ defined by

$$Spin(\mathfrak{q}) \overset{\text{def.}}{=} \{u = v_1\cdots v_k \mid v_j \in \mathfrak{q},\ u\cdot u^* = 1,\ \alpha(u)\gamma(\mathfrak{q})u^* = \gamma(\mathfrak{q}),\ k \text{ even}\}.$$

Observe that the Lie algebra $\mathfrak{spin}(\mathfrak{q})$ of $Spin(\mathfrak{q})$ is [5, Theorem 6.3.6]

$$\mathfrak{spin}(\mathfrak{q}) = \Phi(\mathfrak{so}(\mathfrak{q})),$$

so that

$$\sigma^{\pm}(\Phi(X)) \overset{\text{def.}}{=} \tilde{\gamma}(\Phi(X))_{|_{S_{\mathfrak{q}}^{\pm}}}, \quad X \in \mathfrak{so}(\mathfrak{q})$$

defines a representation of $\mathfrak{spin}(\mathfrak{q})$ in $S_{\mathfrak{q}}^{\pm}$. The representations $(\sigma^+, S_{\mathfrak{q}}^+)$ and $(\sigma^-, S_{\mathfrak{q}}^-)$ are called the half-spin representations of $\mathfrak{spin}(\mathfrak{q})$.

Now we describe the weights of the half-spin representations of $\mathfrak{spin}(\mathfrak{q})$.

Let $\{e_j\}$ and $\{e_j^*\}$ be bases of V and V^*, respectively, such that

$$\langle e_j, e_k\rangle_{\mathfrak{q}} = \langle e_j^*, e_k^*\rangle_{\mathfrak{q}} = 0, \quad \forall j, k,$$

and

$$\langle e_j, e_k^*\rangle_{\mathfrak{q}} = \pm\delta_{jk}. \tag{3.2}$$

A special choice for V and V^* is given below, then the signs in the preceeding formula are given in Lemma 6.1. It is easy to see that the algebra

$$\mathscr{C} = span\left\{R_{e_j,e_j^*}\ \middle|\ 1 \leqslant j \leqslant \frac{dim(\mathfrak{q})}{2}\right\} \tag{3.3}$$

is a Cartan subalgebra of $\mathfrak{so}(\mathfrak{q})$, and $\Phi(\mathscr{C})$ is a Cartan subalgebra of $\mathfrak{spin}(\mathfrak{q})$. Observe that

$$\Phi(\langle e_j, e_j^*\rangle_{\mathfrak{q}} R_{e_j,e_j^*}) = \langle e_j, e_j^*\rangle_{\mathfrak{q}}\,\iota(e_j)\varepsilon(e_j^*) - \tfrac{1}{2}.$$

For each integer l in $\{1, \ldots, \frac{dim(\mathfrak{q})}{2}\}$ and each set $I = \{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_l \leqslant \frac{dim(\mathfrak{q})}{2}\}$, define the element

$$u_I = e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots \wedge e_{i_l}^*,$$

with $u_\emptyset = 1$. Then one obtains

$$\sigma^{\pm}(\Phi(\langle e_j, e_j^* \rangle_{\mathfrak{q}} R_{e_j,e_j^*}))u_I = \begin{cases} -\frac{1}{2}u_I, & j \in I, \\ \frac{1}{2}u_I, & j \notin I, \end{cases}$$

so u_I is a weight vector of weight

$$\lambda_I = \frac{1}{2}\left(\sum_{j \notin I} \varepsilon_j - \sum_{j \in I} \varepsilon_j\right), \tag{3.4}$$

where

$$\varepsilon_j(\Phi(\langle e_k, e_k^* \rangle_{\mathfrak{q}} R_{e_k,e_k^*})) = \delta_{jk}. \tag{3.5}$$

Observe that each such weight has multiplicity one, and there are only two dominant weights, namely $\lambda_\emptyset$ and $\lambda_{\left\{\frac{dim(\mathfrak{q})}{2}\right\}}$. Actually, σ^+ (resp., σ^-) is an irreducible highest weight representation of $\mathfrak{spin}(\mathfrak{q})$ with highest weight $\lambda_\emptyset$ (resp., $\lambda_{\left\{\frac{dim(\mathfrak{q})}{2}\right\}}$) with respect to an appropriate positive system of $\mathscr{C}$-roots. Both $\sigma^{\pm}$ lift to representations of the group $Spin(\mathfrak{q})$.

We now consider the 'spin' representations of $\mathfrak{h}$. These are essentially the restrictions of $\sigma^{\pm}$ to $\mathfrak{h} \subset \mathfrak{so}(\mathfrak{q})$. Since $\langle\ ,\ \rangle_{\mathfrak{q}}$ is $Ad(H)$-invariant, we obtain the Lie group homomorphism

$$\zeta : H \to SO(\mathfrak{q}), \quad h \mapsto Ad(h)_{|_{\mathfrak{q}}},$$

whose differential $d_e\zeta$ at the identity is a homomorphism of Lie algebras:

$$d_e\zeta = ad : \mathfrak{h} \to \mathfrak{so}(\mathfrak{q}).$$

For the above construction we have arbitrarily chosen dual isotropic subspaces V and V^*, and a special basis (as in (3.2)). Now we would like to be more specific about these choices. Since $\mathfrak{a} \oplus \mathfrak{t} \subset \mathfrak{h}$, each root space $(\mathfrak{g}^{\mathbb{C}})_\alpha$, for α in $\Delta(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}})$, is σ-stable so that

$$(\mathfrak{g}^{\mathbb{C}})_\alpha = ((\mathfrak{g}^{\mathbb{C}})_\alpha \cap \mathfrak{h}^{\mathbb{C}}) \oplus ((\mathfrak{g}^{\mathbb{C}})_\alpha \cap \mathfrak{q}^{\mathbb{C}}).$$

But, as $(\mathfrak{g}^{\mathbb{C}})_\alpha$ is one dimensional, one has $(\mathfrak{g}^{\mathbb{C}})_\alpha \subset \mathfrak{h}^{\mathbb{C}}$ or $(\mathfrak{g}^{\mathbb{C}})_\alpha \subset \mathfrak{q}^{\mathbb{C}}$. In the first case, we shall say that α is an $\mathfrak{h}$-root of $\mathfrak{g}^{\mathbb{C}}$, while in the second case α is called a $\mathfrak{q}$-root of $\mathfrak{g}^{\mathbb{C}}$. We denote the set of $\mathfrak{q}$-roots by $\Delta(\mathfrak{q}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}})$. As in Lemma 2.2, $\mathfrak{q} = (\mathfrak{m} \cap \mathfrak{q}) \oplus (\mathfrak{n} \cap \mathfrak{q}) \oplus (\bar{\mathfrak{n}} \cap \mathfrak{q})$. Set V equal to the span of the root vectors for positive $\mathfrak{q}$-roots and V^* equal to the span of the root vectors for the negative $\mathfrak{q}$-roots. Then V and V^* are isotropic and the basis in (3.2) may be chosen to consist of root vectors (properly normalized).

We define the half-spin representations of $\mathfrak{h}$ as the representations $(\mathfrak{s}^+, S_{\mathfrak{q}}^+)$ and $(s^-, S_{\mathfrak{q}}^-)$, where

$$s^{\pm} = \sigma^{\pm} \circ \Phi \circ (d_e\zeta).$$

The description of the set $\Pi(S_{\mathfrak{q}}^{\pm})$ of weights of the half-spin representations of $\mathfrak{h}$ is as follows. As an $\mathfrak{h}$-representation the set of weights of $\mathfrak{q}^{\mathbf{C}}$ relative to $(\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}}$ is precisely the set of $\mathfrak{q}$-roots of $\mathfrak{g}^{\mathbf{C}}$. By definition, $d_e\zeta$ composed with the standard representation of $\mathfrak{so}(\mathfrak{q})$ is exactly the adjoint action of $\mathfrak{h}$ on $\mathfrak{q}$. We may assume that

$$d_e\zeta((\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}})\subset\mathscr{C},$$

so that the pullback $(d_e\zeta)^*$ of $d_e\zeta$ is a bijection between the set of weights relative to $\mathscr{C}$ of the standard representation of $\mathfrak{so}(\mathfrak{q})$ on $\mathfrak{q}^{\mathbf{C}}$, and $\Delta(\mathfrak{q}^{\mathbf{C}}, (\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}})$. Enumerating the roots in $\Delta^+(\mathfrak{q}^{\mathbf{C}}, (\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}})$ as

$$\{\alpha_1, \alpha_2, \dots, \alpha_m\}, \quad \text{with } m = \frac{dim(\mathfrak{q})}{2}$$

we see that the sets $\Pi(S_{\mathfrak{q}}^{\pm})$ of weights of the half-spin representations $S_{\mathfrak{q}}^{\pm}$ of $\mathfrak{h}$ are

$$\Pi(S_{\mathfrak{q}}^{+}) = \{\tfrac{1}{2}(\pm\alpha_1 \pm \alpha_2 \pm \cdots \pm \alpha_{\frac{dim(\mathfrak{q})}{2}}) \mid \text{even number of minuses}\}$$

and

$$\Pi(S_{\mathfrak{q}}^{-}) = \{\tfrac{1}{2}(\pm\alpha_1 \pm \alpha_2 \pm \cdots \pm \alpha_{\frac{dim(\mathfrak{q})}{2}}) \mid \text{odd number of minuses}\}.$$

In particular, each such weight is of multiplicity one. It is useful to observe that one can rewrite these weights as follows. Let F be any subset of $\Delta^+(\mathfrak{q}^{\mathbf{C}}, (\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}})$ and denote by $\langle F \rangle$ the sum of the elements of F. Let $\rho(\mathfrak{q})$ be the half-sum of positive $\mathfrak{q}$-roots of $\mathfrak{g}^{\mathbf{C}}$:

$$\rho(\mathfrak{q}) = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{q}^{\mathbf{C}}, (\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}})} \alpha.$$

Then any weight of $S_{\mathfrak{q}}^{+}$ (resp., $S_{\mathfrak{q}}^{-}$) under $\mathfrak{h}$ is of the form $\rho(\mathfrak{q}) - \langle F \rangle$ for some subset F of $\Delta^+(\mathfrak{q}^{\mathbf{C}}, (\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}})$ of even (resp., odd) cardinality. We may write

$$\Pi(S_{\mathfrak{q}}^{+}) = \{\rho(\mathfrak{q}) - \langle F \rangle \mid F \subset \Delta^+(\mathfrak{q}^{\mathbf{C}}, (\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}}) \text{ and } \#F \text{ even}\}$$

and

$$\Pi(S_{\mathfrak{q}}^{-}) = \{\rho(\mathfrak{q}) - \langle F \rangle \mid F \subset \Delta^+(\mathfrak{q}^{\mathbf{C}}, (\mathfrak{a}\oplus\mathfrak{t})^{\mathbf{C}}) \text{ and } \#F \text{ odd}\}.$$

Let (τ_μ, E_μ) be an irreducible highest weight representation of $\mathfrak{h}$ with highest weight μ. Then the tensor products $S^{\pm}_{\mathfrak{q}} \otimes E_\mu$ are $\mathfrak{h}$-modules under the actions $s^{\pm} \otimes \tau_\mu$. In particular, any weight occurring in the decomposition into irreducibles of the $\mathfrak{h}$-modules $S^{\pm}_{\mathfrak{q}} \otimes E_\mu$ is of the form $\mu + \rho(\mathfrak{q}) - \langle F \rangle$. We assume that our weight μ satisfies the following inequalities. For each simple root $\alpha \in \Delta^+(\mathfrak{h}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}})$,

$$\begin{gathered} \langle \mu + \rho(\mathfrak{q}) - \langle F \rangle + \rho(\mathfrak{h}), \alpha \rangle \geqslant 0, \quad \text{for every } F \subset \Delta^+(\mathfrak{q}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}}), \\ \langle \mu + \rho(\mathfrak{q}) - 2\rho(\mathfrak{m} \cap \mathfrak{k} \cap \mathfrak{q}) + \rho(\mathfrak{h}), \alpha \rangle > 0. \end{gathered} \tag{3.6}$$

Then the weight $\mu + \rho(\mathfrak{q}) - \langle F \rangle + \rho(\mathfrak{h})$ is dominant, relative to $\Delta^+(\mathfrak{h}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}})$, for all subsets F of $\Delta^+(\mathfrak{q}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}})$. Thus, applying the Steinberg formula [7, pp. 111, 112], the decomposition of the $\mathfrak{h}$-modules $S^{\pm}_{\mathfrak{q}} \otimes E_\mu$ into irreducible $\mathfrak{h}$-modules is as follows:

$$S^{\pm}_{\mathfrak{q}} \otimes E_\mu = \sum_{F \subset \mathscr{F}_{\pm}} E_{\mu + \rho(\mathfrak{q}) - \langle F \rangle}, \tag{3.7}$$

where

$$\mathscr{F}_{\pm} = \{F \subset \Delta^+(\mathfrak{q}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}}) : \#F \text{ is even (or odd) and } \mu + \rho(\mathfrak{g}) - \langle F \rangle \text{ is dominant regular}\}.$$

We will use the following technical lemma in the proof of Theorem 6.2. There we will be considering the spin representations of $\mathfrak{m} \cap \mathfrak{h}$. Observe that the killing form restricted to $\mathfrak{m} \cap \mathfrak{q}$ is nondegenerate, so we may construct spin representations of $\mathfrak{m} \cap \mathfrak{h}$ on $S^{\pm}_{\mathfrak{m} \cap \mathfrak{q}}$ in the same manner in which we constructed the spin representations of $\mathfrak{h}$ on $S^{\pm}_{\mathfrak{q}}$ above. We choose the dual isotropic subspaces to be spanned by root vectors for positive (resp., negative) $\mathfrak{m} \cap \mathfrak{q}$-roots. Therefore there is a natural embedding $S^{\pm}_{\mathfrak{m} \cap \mathfrak{q}} \subset S^{\pm}_{\mathfrak{q}}$ as $\mathfrak{m} \cap \mathfrak{h}$-representations. For any irreducible $\mathfrak{h}$-representation E_μ of highest weight μ we set $U_{\mu_{\mathfrak{t}}} = (E_\mu)^{\mathfrak{n} \cap \mathfrak{h}}$, an irreducible representation of $\mathfrak{m} \cap \mathfrak{h}$ on $\mathfrak{h}$ of highest weight $\mu_{\mathfrak{t}} \overset{\text{def.}}{=} \mu|_{\mathfrak{t}}$.

Lemma 3.8. *Consider $S^{\pm}_{\mathfrak{m} \cap \mathfrak{q}} \subset S^{\pm}_{\mathfrak{q}}$ as above. Set*

$$V \overset{\text{def.}}{=} (E_{\mu + \rho(\mathfrak{q}) - 2\rho(\mathfrak{m} \cap \mathfrak{k} \cap \mathfrak{q})})^{\mathfrak{n} \cap \mathfrak{h}}.$$

V is seen to be a constituent in $S^{+}_{\mathfrak{q}} \otimes E_\mu$ by taking $F = \Delta^+(\mathfrak{m} \cap \mathfrak{k} \cap \mathfrak{q})$ in (3.7). *Then the following hold*:

(a) $S^{\pm}_{\mathfrak{m} \cap \mathfrak{q}} \subset (S^{\pm}_{\mathfrak{q}})^{\mathfrak{n} \cap \mathfrak{h}}$.
(b) $V \subset S^{\pm}_{\mathfrak{m} \cap \mathfrak{q}} \otimes U_{\mu_{\mathfrak{t}}}$.

Proof. (a) By Goodman and Wallach [5, Eq. (6.2.1)], as $\mathfrak{h}$-representations $\wedge^2 \mathfrak{q} \cong \mathfrak{so}(\mathfrak{q})$ via $X \wedge Y \mapsto R_{X,Y}$.

Claim. $ad(\mathfrak{n}\cap\mathfrak{h})$ is spanned by $R_{X,Y}$ with X and Y root vectors for $\mathfrak{q}$-roots α and β with $\alpha+\beta$ positive and nonzero on $\mathfrak{a}$.

By the isomorphism above the weights in $\mathfrak{so}(\mathfrak{q})$ are all of the form $\alpha+\beta$ for some $\mathfrak{q}$-roots α and β. The weighs occurring in $\mathfrak{n}\cap\mathfrak{h}$ satisfy $\alpha+\beta>0$ and $(\alpha+\beta)|_{\mathfrak{a}}\neq 0$. The claim follows.

In particular, for such X and Y there are three possibilities:

(i) $X, Y\in\mathfrak{n}\cap\mathfrak{q}$,
(ii) $X\in\mathfrak{n}\cap\mathfrak{q}$, $Y\in\bar{\mathfrak{n}}\cap\mathfrak{q}$ and $\alpha+\beta>0$,
(iii) $X\in\mathfrak{m}\cap\mathfrak{q}$, $Y\in\mathfrak{n}\cap\mathfrak{q}$.

Let $u\in S^{\pm}_{\mathfrak{m}\cap\mathfrak{q}}$. We check that $s^{\pm}(R_{X,Y})(u)=\frac{1}{2}(\gamma(X)\gamma(Y)-\gamma(Y)\gamma(X))(u)=0$ for each of the possibilities. For (i) this is clear since $\gamma(X)=\iota(X)$ and $\gamma(Y)=\iota(Y)$ for $X, Y\in\mathfrak{n}\cap\mathfrak{q}$, and $\iota(X)(u)=\iota(Y)(u)=0$, since $\mathfrak{n}\cap\mathfrak{q}\perp\mathfrak{m}\cap\mathfrak{q}$.

For (ii), again $\gamma(X)=\iota(X)$ is zero on u and

$$\begin{aligned}
s^{\pm}(R_{X,Y})(u) &= \tfrac{1}{2}(\iota(X)\varepsilon(Y)-\varepsilon(Y)\iota(X))(u)\\
&= \tfrac{1}{2}\iota(X)\varepsilon(Y)(u)\\
&= \tfrac{1}{2}\iota(X)(Y\wedge u)\\
&= \tfrac{1}{2}\langle X, Y\rangle_{\mathfrak{q}}u-\tfrac{1}{2}Y\wedge\iota(X)u\\
&= 0,
\end{aligned}$$

since $X\perp Y$ (as $\alpha\neq-\beta$). For (iii),

$$\begin{aligned}
s^{\pm}(R_{X,Y})(u) &= \tfrac{1}{2}(\gamma(X)\iota(Y)-\iota(Y)\gamma(X))(u)\\
&= -\tfrac{1}{2}\iota(Y)\gamma(X)u\\
&= 0,
\end{aligned}$$

since $\gamma(X)(u)\in S^{\pm}_{\mathfrak{m}\cap\mathfrak{q}}$.

(b) It will be enough to show that the highest weight vector in $E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})}$ lies in $S^{+}_{\mathfrak{m}\cap\mathfrak{q}}\otimes U_{\mu_{\mathfrak{t}}}$.

As in (3.7) $S^{+}_{\mathfrak{m}\cap\mathfrak{q}}\otimes U_{\mu_{\mathfrak{t}}}$ contains the irreducible $\mathfrak{m}\cap\mathfrak{h}$-representation of highest weight $\mu_{\mathfrak{t}}+\rho(\mathfrak{m}\cap\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})$. Denote by u_+ the highest weight vector of this constituent.

Claim. Viewing u_+ as in $S^{+}_{\mathfrak{q}}\otimes E_{\mu}$, u_+ is a highest weight vector for the irreducible $\mathfrak{h}$-representation $E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})}\subset S^{+}_{\mathfrak{q}}\otimes E_{\mu}$.

Since u_+ is a highest weight vector for $\mathfrak{m}\cap\mathfrak{h}$ and is killed by $\mathfrak{n}\cap\mathfrak{h}$, u_+ is a highest weight vector for an $\mathfrak{h}$-subrepresentation of $S^+_{\mathfrak{q}}\otimes E_\mu$. We need to check that it has weight $\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})$. The $\mathfrak{t}$-weight is correct. To see this note that by the choice of $\Delta^+(\mathfrak{g}^{\mathbb{C}},(\mathfrak{a}\oplus\mathfrak{t})^{\mathbb{C}})$ following Lemma 2.2, $\rho(\mathfrak{q})|_{\mathfrak{t}}=\rho(\mathfrak{m}\cap\mathfrak{q})$. Therefore,

$$(\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}))|_{\mathfrak{t}}=(\mu+\rho(\mathfrak{m}\cap\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}))|_{\mathfrak{t}}. \tag{3.9}$$

We must calculate the $\mathfrak{a}$-weight of u_+. For any $\mathfrak{t}$-weight vector $u\in S^+_{\mathfrak{m}\cap\mathfrak{q}}\otimes U_{\mu_{\mathfrak{t}}}$ we may write $u=u_F\otimes\tilde{u}$ where $u_F=X_{-\alpha_1}\wedge\cdots\wedge X_{-\alpha_j}$, with $F=\{\alpha_i\}\subset\Delta(\mathfrak{m}\cap\mathfrak{h})$, and $\tilde{u}$ are $\mathfrak{t}$-weight vectors. However, by (3.4) u_F has $\mathfrak{a}\oplus\mathfrak{t}$-weight $(\rho(\mathfrak{q})-\langle F\rangle)|_{\mathfrak{a}}$. This weight is $(\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}))|_{\mathfrak{a}}$, as roots in $\mathfrak{m}$ restrict to 0 on $\mathfrak{a}$. Since $\tilde{u}\in(E_\mu)^{\mathfrak{n}\cap\mathfrak{h}}$, the $\mathfrak{a}$-weight of $\tilde{u}$ is $\mu|_{\mathfrak{a}}$. Therefore, the $\mathfrak{a}$-weight of u is $(\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}))|_{\mathfrak{a}}$. □

Finally, we need to make the following integrability assumption.

Assumption 3.10. *The $\mathfrak{h}$-representations $s^\pm\otimes\tau_\mu$ on $S^\pm_{\mathfrak{q}}\otimes E_\mu$ lift to representations of H.*

As H is connected, the decomposition in (3.7) holds as H-representations.

4. The twisted Dirac operator on G/H and its square

Let (τ_1,E_1) and (τ_2,E_2) be two finite-dimensional representations of H. Denote the corresponding homogeneous vector bundles on G/H by $\mathscr{E}_1\to G/H$ and $\mathscr{E}_2\to G/H$. The spaces of smooth sections are

$$C^\infty(G/H,\mathscr{E}_j)=\{f:G\to E_j|f(gh)=\tau_j(h^{-1})f(g),\ \text{for all } g\in G,\ h\in H\}. \tag{4.1}$$

Equivariant differential operators $C^\infty(G/H,\mathscr{E}_1)\to C^\infty(G/H,\mathscr{E}_2)$ may be described as follows. Let $\mathscr{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and $Hom(E_1,E_2)$ the space of complex linear maps from E_1 to E_2. Then H acts on the tensor product $\mathscr{U}(\mathfrak{g})\otimes Hom(E_1,E_2)$ by

$$h\cdot(u\otimes T)=Ad(h)(u)\otimes(\tau_2(h)\circ T\circ\tau_1^{-1}(h)),\quad \text{for } h\in H,\ u\otimes T\in\mathscr{U}(\mathfrak{g})\otimes Hom(E_1,E_2),$$

where Ad is the adjoint action of G on $\mathfrak{g}$, naturally extended to $\mathscr{U}(\mathfrak{g})$. Then any $\sum u_j\otimes T_j\in\{\mathscr{U}(\mathfrak{g})\otimes Hom(E_1,E_2)\}^H$ (the H-invariants) defines a G-equivariant differential operator

$$Df\overset{\text{def.}}{=}\left(\sum R(u_j)\otimes T_j\right)(f)\overset{\text{def.}}{=}\sum T_j(R(u_j)f).$$

Here $R(u)$ is the right action of $u\in\mathscr{U}(\mathfrak{g})$ extending the action of $\mathfrak{g}$ by left invariant vector fields:

$$R(X)(f)(g)=\frac{d}{dt}f(g\exp(tX))_{|t=0},\quad \forall g\in G,\ \ X\in\mathfrak{g}.$$

For the Dirac operator we fix a basis $\{X_j\}$ of $\mathfrak{q}$ satisfying

$$\langle X_j, X_k\rangle_{\mathfrak{q}} = \pm\delta_{jk}. \tag{4.2}$$

Note that $\langle X_j, X_j\rangle = 1$ (resp., -1) when $X_j\in\mathfrak{q}\cap\mathfrak{s}$ (resp., $\mathfrak{q}\cap\mathfrak{k}$). We also fix an irreducible $\mathfrak{h}$-representation (τ_μ, E_μ) of highest weight $\mu\in(\mathfrak{a}\oplus\mathfrak{t})^*$ satisfying Assumption 3.10. Then by the invariance of $\langle\ ,\ \rangle_{\mathfrak{q}}$ under H

$$\sum_{j=1}^{dim(\mathfrak{q})}\langle X_j, X_j\rangle_{\mathfrak{q}} X_j\otimes\gamma(X_j)\otimes 1 \tag{4.3}$$

is an H-invariant in $\mathscr{U}(\mathfrak{g})\otimes Hom(S_{\mathfrak{q}}^{\pm}\otimes E_\mu, S_{\mathfrak{q}}^{\mp}\otimes E_\mu)$. Therefore, there are well defined G-equivariant differential operators

$$D^{\pm}: C^\infty(G/H, \mathscr{S}_{\mathfrak{q}}^{\pm}\otimes\mathscr{E}_\mu)\to C^\infty(G/H, \mathscr{S}_{\mathfrak{q}}^{\mp}\otimes\mathscr{E}_\mu)$$

defined by

$$D^{\pm} = \sum_{j=1}^{dim(\mathfrak{q})}\langle X_j, X_j\rangle_{\mathfrak{q}} R(X_j)\otimes\gamma(X_j)\otimes 1. \tag{4.4}$$

These are our 'twisted Dirac' operators (see [9, Proposition 3.2] for the case of riemannian symmetric spaces, i.e., for $\sigma=\theta$). Note that the expression in (4.3) is independent of basis satisfying (4.2).

Now let $\langle\ ,\ \rangle_{\mathfrak{h}}$ be the restriction to $\mathfrak{h}$ of the killing form of $\mathfrak{g}$. Let $\{Y_j\}$ be a basis of $\mathfrak{h}$ so that

$$\langle Y_j, Y_k\rangle_{\mathfrak{h}} = \pm\delta_{jk},\quad \forall j,k.$$

Then the Casimir elements Ω_G of G and Ω_H of H are, respectively, defined by

$$\Omega_H = \sum_{j=1}^{dim(\mathfrak{h})}\langle Y_j, Y_j\rangle_{\mathfrak{h}} Y_j^2 \quad\text{and}\quad \Omega_G=\Omega_H+\sum_{j=1}^{dim(\mathfrak{q})}\langle X_j, X_j\rangle_{\mathfrak{q}} X_j^2.$$

Proposition 4.5. *The 'square' of the Dirac operator is*

$$D^{\mp}\circ D^{\pm} = \tfrac{1}{2}(R(\Omega_G)\otimes 1\otimes 1 + 1\otimes s^{\pm}(\Omega_H)\otimes 1 - 1\otimes 1\otimes\tau_\mu(\Omega_H))$$

acting on $\{C^\infty(G)\otimes S_{\mathfrak{q}}^{\pm}\otimes E_\mu\}^H\cong C^\infty(G/H, \mathscr{S}_{\mathfrak{q}}^{\pm}\otimes\mathscr{E}_\mu)$.

Proof. We have

$$D^{\mp}\circ D^{\pm} = \frac{1}{2}\sum_j \langle X_j, X_j\rangle_{\mathfrak{q}} R(X_j^2)\otimes 1\otimes 1 + \frac{1}{2}\sum_{i,j} \langle X_i, X_i\rangle_{\mathfrak{q}}\langle X_j, X_j\rangle_{\mathfrak{q}} R([X_i, X_j])\otimes\gamma(X_i)\gamma(X_j)\otimes 1.$$

Decomposing $[X_i, X_j]$ in the above basis $\{Y_j\}$ of $\mathfrak{h}$, we see that

$$\begin{aligned} D^{\mp}\circ D^{\pm} = & \frac{1}{2}\sum_j \langle X_j, X_j\rangle_{\mathfrak{q}} R(X_j^2)\otimes 1\otimes 1 \\ & + \frac{1}{2}\sum_k \sum_{i,j} \langle [X_i, X_j], Y_k\rangle_{\mathfrak{h}}\langle Y_k, Y_k\rangle_{\mathfrak{h}}\langle X_i, X_i\rangle_{\mathfrak{q}}\langle X_j, X_j\rangle_{\mathfrak{q}} R(Y_k) \\ & \otimes\gamma(X_i)\gamma(X_j)\otimes 1. \end{aligned}$$

On the other hand, as for the Lie algebra $\mathfrak{so}(\mathfrak{q})$, one can also embed $\mathfrak{h}$ in $Cl_2(\mathfrak{q}^{\mathbb{C}})$ using the map $\Phi\circ\zeta$ so that

$$(\Phi\circ\zeta)(Y) = \sum_{i<j} c_{ij}(Y)X_iX_j, \quad \forall Y\in\mathfrak{h}.$$

To determine the coefficients $c_{ij}(Y)$, observe that

$$[X_iX_j, X_k] = \langle X_j, X_k\rangle_{\mathfrak{q}} X_i - \langle X_i, X_k\rangle_{\mathfrak{q}} X_j$$

thus

$$c_{ij}(Y) = \langle X_i, X_i\rangle_{\mathfrak{q}}\langle X_j, X_j\rangle_{\mathfrak{q}}\langle [Y, X_j], X_i\rangle_{\mathfrak{q}}.$$

Hence one obtains that

$$s^{\pm}(Y) = \frac{1}{2}\sum_{i,j} \langle X_i, X_i\rangle_{\mathfrak{q}}\langle X_j, X_j\rangle_{\mathfrak{q}}\langle [Y, X_j], X_i\rangle_{\mathfrak{q}}\gamma(X_i)\gamma(X_j),$$

because $\langle [Y, X_i], X_j\rangle_{\mathfrak{q}} X_iX_j = \langle [Y, X_j], X_i\rangle_{\mathfrak{q}} X_jX_i$. In particular, by invariance of the killing form of $\mathfrak{g}$, $\langle [X_i, X_j], Y_k\rangle_{\mathfrak{h}} + \langle X_j, [X_i, Y_k]\rangle_{\mathfrak{q}} = 0$, so that

$$D^{\mp}\circ D^{\pm} = \frac{1}{2}\sum_j \langle X_j, X_j\rangle_{\mathfrak{q}} R(X_j^2)\otimes 1\otimes 1 - \sum_k \langle Y_k, Y_k\rangle_{\mathfrak{h}} R(Y_k)\otimes s^{\pm}(Y_k)\otimes 1.$$

Finally, since

$$(R\otimes s^{\pm}\otimes 1)(Y_k^2) = 1\otimes s^{\pm}(Y_k^2)\otimes 1 + R(Y_k^2)\otimes 1\otimes 1 + 2R(Y_k)\otimes s^{\pm}(Y_k)\otimes 1, \quad \forall k,$$

one gets that

$$D^{\mp}\circ D^{\pm} = \frac{1}{2}\Big(R(\Omega_G)\otimes 1\otimes 1 + 1\otimes s^{\pm}(\Omega_H)\otimes 1 - \sum_k \langle Y_k, Y_k\rangle_{\mathfrak{h}}(R\otimes s^{\pm}\otimes 1)(Y_k^2)\Big).$$

The conclusion comes from the identity

$$(R\otimes s^{\pm}\otimes 1)(\Omega_H)_{|_{(C^{\infty}(G)\otimes s_{\mathfrak{q}}^{\pm}\otimes E_\mu)^H}} = (1\otimes 1\otimes \tau_\mu)(\Omega_H)_{|_{(C^{\infty}(G)\otimes s_{\mathfrak{q}}^{\pm}\otimes E_\mu)^H}}. \qquad \square$$

5. Construction of the intertwining map

We continue with an irreducible representation (τ_μ, E_μ) of $\mathfrak{h}$ with highest weight μ satisfying (3.6) and (3.10). The space of smooth sections $C^{\infty}(G/H, \mathscr{S}_{\mathfrak{q}}^{\pm}\otimes\mathscr{E}_\mu)$ is as in (4.1). Given a representation $\delta\otimes e^{\nu+\rho_{\mathfrak{g}}}\otimes 1$ on $W\otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}}$ of the parabolic subgroup $P = MAN$, the corresponding homogeneous bundle is denoted by $\mathscr{W}\otimes C_{\nu+\rho_{\mathfrak{g}}}$. The space $C^{\infty}(G/P, \mathscr{W}\otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})$ of smooth sections of this bundle is as in (2.3).

Our goal is to determine the 'parameters' δ and ν for which there is a nonzero G-intertwining map

$$\mathscr{P} : C^{\infty}(G/P, \mathscr{W}\otimes C_{\nu+\rho_{\mathfrak{g}}}) \to C^{\infty}(G/H, \mathscr{S}_{\mathfrak{q}}^{+}\otimes\mathscr{E}_\mu), \tag{5.1}$$

so that the image of $\mathscr{P}$ lies in the kernel of D^{+}. We will also write the intertwining operator as an explicit integral formula.

In the remainder of this section we will determine the parameters δ and ν, and write down a formula for $\mathscr{P}$. In Section 6, we will see that the image of $\mathscr{P}$ lies in the kernel of the Dirac operator.

We begin with a proposition which gives a condition on the infinitesimal character. Let $\rho_{\mathfrak{h}}$ be half the sum of the positive (restricted) $\mathfrak{a}$-roots in $\mathfrak{h}$. We now specify a positive system of $\mathfrak{t}$-roots in $\mathfrak{m}$ by $\Delta^{+}(\mathfrak{m},\mathfrak{t}) = \{\alpha\colon \langle\mu,\alpha\rangle > 0\}$. We obtain the following conditions on the parameters of $\pi_{\delta,\nu}$.

Proposition 5.2. *Let* (τ_μ, E_μ) *be an irreducible highest weight representation of* $\mathfrak{h}$ *with highest weight* μ, *where* μ *satisfies conditions* (3.6) *and* (3.10). *Assume there exists a* *nonzero intertwining operator* $\mathscr{P}^{\pm}$ *from the G-module* $C^{\infty}(G/P, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})$ *into the G-module* $C^{\infty}(G/H, \mathscr{S}^{\pm}_{\mathfrak{q}} \otimes \mathscr{E}_{\mu})$ *such that* $D^{\pm} \circ \mathscr{P}^{\pm} = 0$, *then*

$$||char(\delta) + \nu|| = ||\mu + \rho(\mathfrak{h})||,$$

where $char(\delta)$ *is the infinitesimal character of the discrete series representation* δ *of* M.

Proof. Since $D^{\pm} \circ \mathscr{P}^{\pm} = 0$, we have $D^{\mp} \circ D^{\pm} \circ \mathscr{P}^{\pm} = 0$. Then, using Proposition 4.5, one obtains that

$$||char(\delta) + \nu||^2 - ||\rho(\mathfrak{g})||^2 + ||\rho(\mathfrak{q}) + \rho(\mathfrak{h})||^2$$
$$- ||\rho(\mathfrak{h})||^2 - ||\mu + \rho(\mathfrak{h})||^2 + ||\rho(\mathfrak{h})||^2 = 0.$$

Observe that even though the representations $S^{\pm}_{\mathfrak{q}}$ of $\mathfrak{h}$ are not in general irreducible, the Casimir operator of H acts on each irreducible component by the same scalar. This follows from [9, Lemma 2.2] and the 'Weyl unitary trick'. We therefore have

$$||char(\delta) + \nu|| = ||\mu + \rho(\mathfrak{h})||. \qquad \square$$

In order to specify the representation δ we need to carefully describe the discrete series representations of M. As noted earlier M is a cuspidal parabolic subgroup, i.e., M and $M \cap K$ have equal (complex) ranks, so M has nonempty discrete series. The fact that M is generally neither semisimple nor connected complicates the description, see [6, Section 27; 7, Chapter XII, Section 8] for details.

First, the identity component M_0 has compact center and there is a subgroup $F \subset Z_M$ (the center of M) so that $M^{\#} \overset{\text{def.}}{=} M_0 \cdot F$ has discrete series representations parameterized by the data

(a) $\lambda \in \mathfrak{t}^*$ with $\lambda - \rho(\mathfrak{m})$ analytically integral, and
(b) a character χ of F so that $\chi = e^{\lambda - \rho(\mathfrak{m})}$ on $M_0 \cap F$.

When λ is a dominant $\mathfrak{t}$-weight the corresponding discrete series representation, denoted by $\pi(\lambda, \chi; M^{\#})$, satisfies the following:

(i) $\pi(\lambda, \chi; M^{\#})$ has infinitesimal character λ.
(ii) The minimal $K \cap M^{\#}$-type has highest weight $\lambda + \rho(\mathfrak{m}) - 2\rho(\mathfrak{m} \cap \mathfrak{k})$.
(iii) The action of F on $\pi(\lambda, \chi; M^{\#})$ is by the character χ.

Finally, the representations

$$\pi(\lambda, \chi; M) = Ind^{M}_{M^{\#}}(\pi(\lambda, \chi; M^{\#}))$$

of M are irreducible and in the discrete series.

Remark 5.3. Set $P^{\#} \stackrel{\text{def.}}{=} M^{\#}AN$. For $\delta = \pi(\lambda, \chi; M), Ind_P^G(\delta \otimes \nu) = Ind_{M^{\#}AN}^G(\pi(\lambda, \chi; M^{\#}) \otimes \nu)$. We find it slightly simpler to find an intertwining map

$$\mathscr{P} : C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_\nu) \to C^\infty(G/H, \mathscr{S}_{\mathfrak{q}}^+ \otimes \mathscr{E}_\mu)$$

rather than as in (5.1). Thus our W will be one of the $\pi(\lambda, \chi; M^{\#})$.

We will denote the restrictions of $(\mathfrak{a} \oplus \mathfrak{t})^{\mathbf{C}}$-weights to $\mathfrak{t}$ and $\mathfrak{a}$ by applying subscripts, therefore $\mu_{\mathfrak{t}}$ (resp., $\mu_{\mathfrak{a}}$) is the restriction of μ to $\mathfrak{t}$ (resp., $\mathfrak{a}$).

Theorem 5.4. *Assume that the weight μ satisfies conditions* (3.6) *and* (3.10). *Consider the discrete series representation (δ, W) of $M^{\#}$ with $\delta = \pi(\mu_{\mathfrak{t}} + \rho(\mathfrak{m} \cap \mathfrak{h}), \chi; M^{\#})$ (χ will be determined below). Let $\nu = \mu_{\mathfrak{a}} + \rho_{\mathfrak{h}}$.*

Then we have

$$Hom_G(C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}}), C^\infty(G/H, \mathscr{S}_{\mathfrak{q}}^+ \otimes \mathscr{E}_\mu)) \neq \{0\}.$$

Proof. First, one has the standard isomorphisms

$$\begin{aligned} &Hom_G(C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}}), C^\infty(G/H, \mathscr{S}_{\mathfrak{q}}^\pm \otimes \mathscr{E}_\mu)) \\ &\quad \simeq Hom_H(C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}}), S_{\mathfrak{q}}^\pm \otimes E_\mu) \\ &\quad \simeq Hom_H((S_{\mathfrak{q}}^\pm \otimes E_\mu)^*, C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})'), \end{aligned} \tag{5.5}$$

where $C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})'$ is the continuous dual of $C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})$. Therefore, our problem is to find certain H-finite vectors in $C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})'$. This is hard in general. However, as in [12], we shall restrict our attention to H-finite vectors in $C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})'$ which are supported on closed H-orbits in $G/P^{\#}$. Observe that by Lemma 2.2, our choice of $P = MAN$ guarantees that $P \cap H$ is a minimal parabolic subgroup of H and $H \cdot eP \cong H/H \cap P$ is a closed H-orbit in G/P. Also $H/P^{\#} \cap H$ is a closed H-orbit in $G/P^{\#}$. The space of distributions in $C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})'$, which restrict to smooth functions on $H \cdot eP^{\#} \simeq H/P^{\#} \cap H$ can be identified with the space $C^\infty(H/P^{\#} \cap H, \mathscr{W}^* \otimes \mathbf{C}_{-\nu-\rho_{\mathfrak{g}}+2\rho_{\mathfrak{h}}})$, as in [12, Lemma 2.3]:

$$C^\infty(H/H \cap P^{\#}, \mathscr{W}^* \otimes \mathbf{C}_{-\nu-\rho_{\mathfrak{g}}+2\rho_{\mathfrak{h}}}) \subset C^\infty(G/P^{\#}, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})'.$$

Moreover, we have the obvious isomorphisms

$$\begin{aligned}
&Hom_H((S_{\mathfrak{q}}^{\pm}\otimes E_\mu)^*, C^\infty(H/H\cap P^\#, \mathscr{W}^*\otimes \mathbf{C}_{-\nu-\rho_{\mathfrak{g}}+2\rho_{\mathfrak{h}}}))\\
&\quad\simeq Hom_{P^\#\cap H}((S_{\mathfrak{q}}^{\pm}\otimes E_\mu)^*, W^*\otimes \mathbf{C}_{-\nu-\rho_{\mathfrak{g}}+2\rho_{\mathfrak{h}}})\\
&\quad\simeq Hom_{P^\#\cap H}(W\otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}-2\rho_{\mathfrak{h}}}, (S_{\mathfrak{q}}^{\pm}\otimes E_\mu))\\
&\quad\simeq Hom_{M\cap H\cdot A}(W\otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}-2\rho_{\mathfrak{h}}}, (S_{\mathfrak{q}}^{\pm}\otimes E_\mu)^{\mathfrak{n}\cap\mathfrak{h}}).
\end{aligned}\tag{5.6}$$

Observe that by formula (3.7), and the fact that $\Delta^+(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})$ has even cardinality, $S_{\mathfrak{q}}^{+}\otimes E_\mu$ contains the irreducible constituent

$$E_{\mu+\rho(\mathfrak{q})-\langle\Delta+(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})\rangle}$$

of highest weight $\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})$. The highest $\mathfrak{a}$-weight is $\mu_{\mathfrak{a}}+\rho_{\mathfrak{q}}$, where $\rho_{\mathfrak{q}}$ is $\rho(\mathfrak{q})_{\mathfrak{a}}$. This forces our choice for ν.

We now consider

$$V\overset{\text{def.}}{=}(E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})})^{\mathfrak{n}\cap\mathfrak{h}}\tag{5.7}$$

as an $\mathfrak{m}\cap\mathfrak{h}$-representation.

Claim. $(\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}))|_{\mathfrak{t}}=\mu_{\mathfrak{t}}+\rho(\mathfrak{m})+\rho(\mathfrak{m}\cap\mathfrak{h})-2\rho(\mathfrak{m}\cap\mathfrak{k})$.

To see this note that by the choice of $\Delta^+(\mathfrak{g}^{\mathrm{C}},(\mathfrak{a}\oplus\mathfrak{t})^{\mathrm{C}})$ following Lemma 2.2, $\rho(\mathfrak{q})|_{\mathfrak{t}}=\rho(\mathfrak{m}\cap\mathfrak{q})$. Therefore

$$\begin{aligned}
&(\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}))|_{\mathfrak{t}}\\
&\quad=\ (\mu+\rho(\mathfrak{m}\cap\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q}))|_{\mathfrak{t}}\\
&\quad=\ \mu_{\mathfrak{t}}+\rho(\mathfrak{m}\cap\mathfrak{q})+2\rho(\mathfrak{m}\cap\mathfrak{h})-2\rho(\mathfrak{m}\cap\mathfrak{h})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})\\
&\quad=\ \mu_{\mathfrak{t}}+\rho(\mathfrak{m})+\rho(\mathfrak{m}\cap\mathfrak{h})-2\rho(\mathfrak{m}\cap\mathfrak{k}).
\end{aligned}$$

The last equality follows from $\mathfrak{m}\cap\mathfrak{h}=\mathfrak{m}\cap\mathfrak{h}\cap\mathfrak{k}$, by Lemma 2.2(a). We conclude that V is the irreducible representation of $M\cap H$ with highest weight $\mu_{\mathfrak{t}}+\rho(\mathfrak{m})+\rho(\mathfrak{m}\cap\mathfrak{h})-2\rho(\mathfrak{m}\cap\mathfrak{k})$.

We choose the character χ of F to be the action of F on V.

Therefore V is a constituent of $\delta|_{M^\#\cap H}$. This proves that

$$Hom_{M^\#\cap H\cdot A}(W\otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}-2\rho_{\mathfrak{h}}}, (S_{\mathfrak{q}}^{+}\otimes E_\mu)^{\mathfrak{n}\cap\mathfrak{h}})\neq\{0\},$$

and the theorem follows from the isomorphisms in (5.5) and (5.6) above. $\square$

We give an explicit formula for the intertwining operator of Theorem 5.4. Indeed, let $t \in Hom_{M^\# \cap H \cdot A}(W \otimes \mathbf{C}_{\nu+\rho_\mathfrak{g}-2\rho_\mathfrak{h}}, (S_\mathfrak{q}^+ \otimes E_\mu)^{\mathfrak{n}\cap\mathfrak{h}})$ be a nonzero element of $Hom_{M^\# \cap H \cdot A}(W \otimes \mathbf{C}_{\nu+\rho_\mathfrak{g}-2\rho_\mathfrak{h}}, (E_{\mu+\rho(\mathfrak{q})-2\rho(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{q})})^{\mathfrak{n}\cap\mathfrak{h}})$. Tracing through the isomorphisms of (5.5) and (5.6), one obtains a nonzero intertwining operator $\mathscr{P}_t$ from the G-module $C^\infty(G/P^\#, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_\mathfrak{g}})$ into the G-module $C^\infty(G/H, \mathscr{S}_\mathfrak{q}^+ \otimes \mathscr{E}_\mu)$ defined as follows:

$$(\mathscr{P}_t\phi)(g) = \int_{H\cap K} [s^+(\ell) \otimes \tau_\mu(\ell)](t(\phi(g\ell)))\, dl, \tag{5.8}$$

for all $\phi \in C^\infty(G/P^\#, \mathscr{W} \otimes \mathbf{C}_{\nu+\rho_\mathfrak{g}})$ and $g \in G$. To obtain a more explicit formula for $\mathscr{P}_t$ we consider an explicit geometric realization of the discrete series representations δ of $M^\#$. By Proposition A.3, W occurs as the kernel of the Dirac operator

$$D^+_{M^\#/M^\#\cap H} : C^\infty(M^\#/M^\# \cap H, \mathscr{S}^+_{\mathfrak{m}\cap\mathfrak{q}} \otimes \mathscr{U}_{\mu_\mathfrak{t}}) \to C^\infty(M^\#/M^\# \cap H, \mathscr{S}^-_{\mathfrak{m}\cap\mathfrak{q}} \otimes \mathscr{U}_{\mu_\mathfrak{t}}).$$

Now, evaluation at e gives a nonzero $M^\# \cap H$-homomorphism

$$C^\infty(M^\#/M^\# \cap H, \mathscr{S}^+_{\mathfrak{m}\cap\mathfrak{q}} \otimes \mathscr{U}_{\mu_\mathfrak{t}}) \to S^+_{\mathfrak{m}\cap\mathfrak{q}} \otimes U_{\mu_\mathfrak{t}}.$$

There is also a nonzero projection

$$\pi : S^+_{\mathfrak{m}\cap\mathfrak{q}} \otimes U_{\mu_\mathfrak{t}} \to V \subset (S^+_\mathfrak{q} \otimes E_\mu)^{\mathfrak{n}\cap\mathfrak{h}}$$

(with V as in Eq. (5.7)). Now take t to be $t(f) \overset{\text{def.}}{=} \pi(f(e))$, which is nonzero by Proposition A.2. With this choice of t we have a nonzero intertwining operator

$$(\mathscr{P}\phi)(g) = \int_{H\cap K} [s^+(\ell) \otimes \tau_\mu(\ell)](\pi(\phi(g\ell))(e))\, d\ell. \tag{5.9}$$

6. Solutions of the Dirac equations

In this section, we show that the image of $\mathscr{P}$ lies in the kernel of the Dirac operator. For this we need a simple lemma.

Lemma 6.1. *Consider* $(\mathfrak{m}\cap\mathfrak{q})^\perp = (\mathfrak{n}\cap\mathfrak{q}) \oplus (\bar{\mathfrak{n}}\cap\mathfrak{q})$. *The subspaces* $\mathfrak{n}\cap\mathfrak{q}$ *and* $\bar{\mathfrak{n}}\cap\mathfrak{q}$ *of* $\mathfrak{q}$ *are isotropic for* $\langle\ ,\ \rangle_\mathfrak{q}$ *and there is a basis* $\{E_k\}$ *of* $\mathfrak{n}\cap\mathfrak{q}$ *and a basis* $\{\bar{E}_k\}$ *of* $\bar{\mathfrak{n}}\cap\mathfrak{q}$ *so that*

$$\langle E_k, \bar{E}_l \rangle_\mathfrak{q} = \delta_{kl}.$$

In particular,

$$\langle E_k \pm \bar{E}_k, E_l \pm \bar{E}_l \rangle_\mathfrak{q} = \pm 2\delta_{kl}$$

and

$$\langle E_k \pm \bar{E}_k, E_l \mp \bar{E}_l \rangle_{\mathfrak{q}} = 0.$$

Proof. This is clear since the killing form is nondegenerate on $(\mathfrak{n}+\bar{\mathfrak{n}})\cap\mathfrak{q}$ and zero on each of $\mathfrak{n}\cap\mathfrak{q}$ and $\bar{\mathfrak{n}}\cap\mathfrak{q}$. $\square$

Theorem 6.2. *For each $\phi \in C^{\infty}(G/P^{\#}, \mathscr{W}\otimes \mathbf{C}_{\nu+\rho(\mathfrak{g})}), D^{+}(\mathscr{P}(\phi)) = 0$.*

Proof. Recalling definition (4.4) of $D^{\pm}$, we have, for all functions ϕ in $C^{\infty}(G/P; \mathscr{W}\otimes \mathbf{C}_{\nu+\rho_{\mathfrak{g}}})$ and for all g in G,

$$\begin{aligned}
&(D^{+}(\mathscr{P}\phi))(g)\\
&\quad = \int_{H\cap K} \sum_{j=1}^{dim(\mathfrak{q})} \langle X_j, X_j \rangle_{\mathfrak{q}} ((R(X_j)\otimes\gamma(X_j)\otimes 1)\ell \cdot \pi(\phi(\cdot\ell)))(e)|_{g}\, d\ell\\
&\quad = \int_{H\cap K} \ell \cdot \sum_{j=1}^{dim(\mathfrak{q})} \langle X_j, X_j \rangle_{\mathfrak{q}} ((R(X_j)\otimes\gamma(X_j)\otimes 1)\pi(\phi(\cdot)))(e)|_{g\ell}\, d\ell
\end{aligned}$$

by the H-invariance of $\sum_{j=1}^{dim(\mathfrak{q})} \langle X_j, X_j \rangle_{\mathfrak{q}} R(X_j)\otimes\gamma(X_j)\otimes 1$. Now note that $\phi(gm)(e) = (m^{-1}\cdot\phi(g))(e) = \phi(g)(m)$. In particular, for $X\in\mathfrak{m}, ((R(X)\phi)(gl))(e) = ((R(X)\phi(g\ell))(e)$. We may rewrite the above in two terms as

$$\begin{aligned}
&\int_{H\cap K} \sum_{X_j\in(\mathfrak{m}\cap\mathfrak{q})^{\perp}} \langle X_j, X_j \rangle_{\mathfrak{q}} \ell\cdot\gamma(X_j)\pi((R(X_j)\phi)(g\ell))(e))\, d\ell\\
&\quad + \int_{H\cap K} \ell\cdot\pi\left(\sum_{X_j\in\mathfrak{m}\cap\mathfrak{q}} \langle X_j, X_j \rangle_{\mathfrak{q}} \gamma(X_j)((R(X_j)\phi(g\ell))(e))\right) d\ell. \qquad (6.3)
\end{aligned}$$

The second term is $\int_{H\cap K} \ell\cdot\pi((D^{+}_{M^{\#}/M^{\#}\cap H}\phi(g\ell))(e))\, d\ell$.

Now recall that the Dirac operator is independent of the basis (subject to (4.2)), so we may choose the X_j's in $(\mathfrak{m}\cap\mathfrak{q})^{\perp}$ to be the vectors $\frac{1}{\sqrt{2}}(E_k \pm \bar{E}_k)$, for $k = 1, 2, \ldots, \frac{1}{2}dim(\mathfrak{m}\cap\mathfrak{q})$. Recall that $\mathfrak{n}\cap\mathfrak{q}\subset V$ and $\bar{\mathfrak{n}}\cap\mathfrak{q}\subset V^{*}$ in the notation of Section 3. Note that

$$\begin{aligned}
&R(E_j + \bar{E}_j)\otimes\gamma(E_j+\bar{E}_j) - R(E_j - \bar{E}_j)\otimes\gamma(E_j - \bar{E}_j)\\
&\quad = 2R(E_j)\otimes\varepsilon(\bar{E}_j) + 2R(\bar{E}_j)\otimes\iota(E_j).
\end{aligned}$$

Therefore, the terms in (6.3) corresponding to $X_j\in(\mathfrak{m}\cap\mathfrak{q})^{\perp}$ contain expressions of the form

$$\varepsilon(\bar{E}_j)\pi(((R(E_j)\phi)(g\ell))(e)) \quad \text{and} \quad \iota(E_j)\pi(((R(\bar{E}_j)\phi)(g\ell))(e)).$$

The first is zero since ϕ is right N-invariant by (2.3). The second vanishes since the image of π is contained in $S^{\pm}_{\mathfrak{m}\cap\mathfrak{q}} \otimes U_{\mu_{\mathfrak{t}}}$ by Lemma 3.8 and the fact that $(\mathfrak{m}\cap\mathfrak{q}) \perp (\mathfrak{n}\cap\mathfrak{q})$.

We conclude that

$$D^{+}(\mathscr{P}\phi)(g) = \int_{H\cap K} \ell \cdot \pi((D^{+}_{M^{\#}/M^{\#}\cap H}\phi(g\ell))(e))\, d\ell.$$

The theorem follows. $\square$

Appendix

In Section 5, we use the realization of the discrete series representations of $M^{\#}$ as solutions of twisted Dirac operators on $M^{\#}/M^{\#}\cap H$. This appendix justifies the use of this realization. We observe that $M^{\#}$ decomposes into a compact and a noncompact factor. As $M^{\#}\cap H$ is compact, σ restricts to a Cartan involution on the noncompact factor. In particular, $M^{\#}/M^{\#}\cap H$ is (locally) a product of riemannian symmetric spaces of compact and noncompact type. We treat the two cases separately.

Therefore, we consider a riemannian symmetric space G/K. Note that this is the situation of Section 2 when $\sigma = \theta$. The twisted Dirac operators are defined as in formula (4.4). Suppose that λ is dominant for $\Delta^{+}(\mathfrak{g},\mathfrak{t})$. (Note that $\mathfrak{a} = 0$ when $\sigma = \theta$.)

Case 1: Suppose that G is compact. Then by Slebarski [10], the irreducible (finite dimensional) representation of G with infinitesimal character λ occurs as the kernel of the twisted Dirac operator

$$D^{+} : L^{2}(G/K, \mathscr{S}^{+}_{\mathfrak{s}} \otimes \mathscr{E}_{\lambda-\rho(\mathfrak{k})}) \to L^{2}(G/K, \mathscr{S}^{-}_{\mathfrak{s}} \otimes \mathscr{E}_{\lambda-\rho(\mathfrak{k})}).$$

Case 2: Suppose that G is noncompact. Then, by Parthasarathy [9] and Atiyah and Schmid [1], the discrete series representation of G with infinitesimal character λ and minimal K-type $\lambda + \rho(\mathfrak{g}) - 2\rho(\mathfrak{k})$ occurs as the kernel of the twisted Dirac operator

$$D^{+} : L^{2}(G/K, \mathscr{S}^{+}_{\mathfrak{s}} \otimes \mathscr{E}_{\lambda-\rho(\mathfrak{k})}) \to L^{2}(G/K, \mathscr{S}^{-}_{\mathfrak{s}} \otimes \mathscr{E}_{\lambda-\rho(\mathfrak{k})}).$$

Remark A.1. Here D^{+} is an elliptic operator, so the L^{2}-solution spaces to $D^{+} = 0$ consist of smooth functions.

In each of the two cases above evaluation at e gives a well-defined nonzero K-homomorphism

$$Eval_{e} : Ker(D^{+}) \to S^{+}_{\mathfrak{s}} \otimes E_{\lambda-\rho(\mathfrak{k})}.$$

Consider this evaluation followed by the projection $\pi : S^{+}_{\mathfrak{s}} \otimes E_{\lambda-\rho(\mathfrak{k})} \to E_{\lambda+\rho(\mathfrak{g})-2\rho(\mathfrak{k})}$. Then we have the following.

Proposition A.2.

$$\pi \circ Eval_e \neq 0, \quad on\ Ker\ D^+.$$

Proof. In the noncompact case this follows from the discussion in [1, Section 5], in particular the paragraph preceding (5.11) and the fact that the K-type $E_{\lambda+\rho(\mathfrak{g})-2\rho(\mathfrak{k})}$ occurs in the discrete series with multiplicity one. In the compact case this follows from the proof of [8, Theorem 4]. $\square$

Let δ be the discrete series representation of M_0 with the infinitesimal character $\mu_{\mathfrak{t}} + \rho(\mathfrak{m} \cap \mathfrak{h})$ and minimal $M_0 \cap K$-type $\mu_{\mathfrak{t}} + \rho(\mathfrak{m} \cap \mathfrak{h}) + \rho(\mathfrak{m}) - 2\rho(\mathfrak{m} \cap \mathfrak{k})$. We may conclude from the two cases above that δ occurs in the kernel of

$$D^+ : L^2(M_0/M_0 \cap H, \mathscr{S}^+_{\mathfrak{m}\cap\mathfrak{q}} \otimes \mathscr{U}_\mu) \to L^2(M_0/M_0 \cap H, \mathscr{S}^-_{\mathfrak{m}\cap\mathfrak{q}} \otimes \mathscr{U}_\mu).$$

To pass to $M^\#$ note that $M^\#/M^\# \cap H \cong M_0/M_0 \cap H$.

Denote by $D^+_{M^\#/M^\# \cap H}$ the Dirac operator for $M^\#/M^\# \cap H$.

Proposition A.3. *The discrete series representation* $\pi(\mu_{\mathfrak{t}} + \rho(\mathfrak{m} \cap \mathfrak{h}), \chi; M^\#)$ *may be realized as the kernel of*

$$D^+_{M^\#/M^\# \cap H} : L^2(M^\#/M^\# \cap H, \mathscr{S}^+_{\mathfrak{m}\cap\mathfrak{q}} \otimes \mathscr{U}_{\mu_{\mathfrak{t}}}) \to L^2(M^\#/M^\# \cap H, \mathscr{S}^-_{\mathfrak{m}\cap\mathfrak{q}} \otimes \mathscr{U}_{\mu_{\mathfrak{t}}}).$$

References

[1] M. Atiyah, W. Schmid, A geometric construction of the discrete series for semisimple Lie groups, Invent. Math. 42 (1977) 1–62.
[2] L. Barchini, Szegö mappings, harmonic forms and Dolbeault cohomology, J. Funct. Anal. 118 (1993) 351–406.
[3] L. Barchini, A.W. Knapp, R. Zierau, Intertwining operators into Dolbeault cohomology representations, J. Funct. Anal. 107 (1992) 302–341.
[4] R.W. Donley, Intertwining operators into cohomology representations for semisimple Lie groups, J. Funct. Anal. 151 (1997) 138–165.
[5] R. Goodman, N.R. Wallach, Representations and Invariants of the Classical Groups, Encyclopedia of Mathematics and its Applications, Vol. 68, Cambridge University Press, Cambridge, 1998.
[6] Harish-Chandra, Harmonic analysis on real reductive groups, J. Funct. Anal. 19 (1975) 102–204.
[7] A.W. Knapp, Representation Theory of Semisimple Groups: An Overview Based on Examples, Princeton University Press, Princeton, NJ, 1986.
[8] G.D. Landweber, Harmonic spinors on homogeneous spaces, Representation Theory 4 (2000) 466–473.
[9] R. Parthasarathy, Dirac operator and discrete series, Ann. of Math. 96 (1972) 1–30.
[10] S. Slebarski, The Dirac operator on homogeneous spaces and representations of reductive Lie groups II, Amer. J. Math. 109 (1987) 499–520.

[11] J.A. Wolf, Partially harmonic spinors and representations of reductive Lie groups, J. Funct. Anal. 15 (2) (1974) 117–154.
[12] R. Zierau, Unitarity of certain Dolbeault cohomology representations, in: M. Eastwood, J.A. Wolf, R. Zierau (Eds.), The Penrose Transform and Analytic Cohomology in Representation Theory, Contemporary Mathematics, Vol. 154, Amer. Math. Soc., Providence, RI, 1993, pp. 239–259.